

# Optimization of Partial Search

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Quantum Grover search algorithm can find a target item in a database faster than any classical algorithm. One can trade accuracy for speed and find a part of the database (a block) containing the target item even faster, this is partial search. A partial search algorithm was recently suggested by Grover and Radhakrishnan. Here we optimize it. Efficiency of the search algorithm is measured by number of queries to the oracle. The author suggests new version of Grover-Radhakrishnan algorithm which uses minimal number of queries to the oracle. The algorithm can run on the same hardware which is used for the usual Grover algorithm.

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## INTRODUCTION

Database search has many applications and used widely. Grover discovered a quantum algorithm that searches faster than a classical algorithm [1]. It consists of repetition of the Grover iteration  $G_1$ . We shall call it global iteration, see (5). The number of repetitions is:

$$j_{\text{full}} = \frac{\pi}{4} \sqrt{N} \quad (1)$$

for a database with large number of entries  $N$ . After  $j_{\text{full}}$  the algorithm finds the target item.

Sometimes it is sufficient to find an approximate location of the target item. A partial search considers the following problem: a database is separated into  $K$  blocks, of a size  $b = N/K$ . We want to find a block with the target item, not the target item itself. First quantum algorithm for a partial search was suggested by Grover and Radhakrishnan in [7]. They showed that classical partial search takes  $\sim (N - b)$  queries, but quantum algorithm takes only  $\sim (\sqrt{N} - \mathbf{c}\sqrt{b})$  queries. Here  $\mathbf{c}$  is a positive coefficient. This algorithm uses several global iterations  $G_1^{j_1}$  and then several local iteration  $G_2^{j_2}$ , see (8). Local searches are made in each block separately in parallel. Here we optimize this algorithm: the number of queries to the oracle minimized, the coefficient  $\mathbf{c}$  is increased. Exact expression for the number of queries necessary to find the target block is given by formulae (17),(20) and (23). Efficiency of search algorithms is measured by number of queries to the oracle, we call it number of iterations. The lower bound is in the end of the paper. Partial search can use the same hardware as the full search.

## PARTIAL SEARCH

### Global Iterations

First let us remind the full Grover search. We consider a database with one target item. The aim of the Grover algorithm is to identify a target state  $|t\rangle$  among an unordered set of  $N$  states. This is achieved by repeating global iteration which is defined in terms of two operators. The first changes the sign of the target state  $|t\rangle$  only:

$$I_t = \hat{I} - 2|t\rangle\langle t|, \quad \langle t|t\rangle = 1, \quad (2)$$

where  $\hat{I}$  is the identity operator. The second operator,

$$I_{s_1} = \hat{I} - 2|s_1\rangle\langle s_1|, \quad (3)$$

changes the sign of the uniform superposition of all basis states  $|s_1\rangle$ ,

$$|s_1\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle, \quad \langle s_1|s_1\rangle = 1. \quad (4)$$

The **global iteration** is defined as a unitary operator

$$G_1 = -I_{s_1} I_t. \quad (5)$$

We shall use eigenvectors of  $G_1$ :

$$G_1 |\psi_1^\pm\rangle = \lambda_1^\pm |\psi_1^\pm\rangle, \quad \lambda_1^\pm = \exp[\pm 2i\theta_1], \quad |\psi_1^\pm\rangle = \frac{1}{\sqrt{2}}|t\rangle \pm \frac{i}{\sqrt{2}} \left( \sum_{\substack{x=0 \\ x \neq t}}^{N-1} \frac{|x\rangle}{\sqrt{(N-1)}} \right). \quad (6)$$

They were found in [4]. The angle  $\theta_1$  is defined by

$$\sin^2 \theta_1 = \frac{1}{N}. \quad (7)$$

### Grover-Radhakrishnan Algorithm for Partial Search

The partial search algorithm is designed to find a block with the target item: the target block. We shall call other blocks: non-target blocks. The algorithm uses  $j_1$  global iteration and  $j_2$  local iterations. Local iterations are Grover iterations for each block:

$$G_2 = -I_{s_2} I_t. \quad (8)$$

$I_t$  is given by (2), but  $I_{s_2}$  is different. The action of the operator  $I_{s_2}$  in an individual block can be represented as:

$$I_{s_2} |_{\text{block}} = \hat{I} |_{\text{block}} - 2|s_2\rangle\langle s_2|, \quad |s_2\rangle = \frac{1}{\sqrt{b}} \sum_{\text{the block}} |x\rangle. \quad (9)$$

For the whole database we should write  $I_{s_2}$  as a direct sum of the operators (9) over all blocks.

Relevant eigenvectors of  $G_2$  are:

$$G_2 |\psi_2^\pm\rangle = \lambda_2^\pm |\psi_2^\pm\rangle, \quad \lambda_2^\pm = \exp[\pm 2i\theta_2], \quad |\psi_2^\pm\rangle = \frac{1}{\sqrt{2}}|t\rangle \pm \frac{i}{\sqrt{2}}|\text{ntt}\rangle \quad (10)$$

Here the  $|\text{ntt}\rangle$  is a normalized sum of all non-target items in the target block:

$$|\text{ntt}\rangle = \frac{1}{\sqrt{b-1}} \sum_{\substack{x \neq t \\ \text{target block}}} |x\rangle, \quad \langle \text{ntt} | \text{ntt} \rangle = 1. \quad (11)$$

We shall need an angle  $\theta_2$  given by

$$\sin^2 \theta_2 = \frac{K}{N} = \frac{1}{b}. \quad (12)$$

Local iteration does not change non-target blocks. Inside the target block it acts similar to the usual Grover search. After several global iterations and several local we still have to apply one more global iteration. The partial search algorithm creates a vector

$$|d\rangle = G_1 G_2^{j_2} G_1^{j_1} |s_1\rangle. \quad (13)$$

In the state  $|d\rangle$  the amplitudes of all items in non-target blocks are the same. Using eigenvectors of local (10) and global iterations from (6) we can calculate this amplitude and require that it vanishes:

$$\begin{aligned} \frac{-N}{\sqrt{N-1}} \left( \frac{1}{2} - \frac{1}{K} \right) \cos((2j_1 + 1)\theta_1) &= \cos(2j_2\theta_2) \sin((2j_1 + 1)\theta_1) + \sqrt{\frac{b-1}{N-1}} \sin(2j_2\theta_2) \cos((2j_1 + 1)\theta_1) \\ &\quad - \sqrt{b-1} \sin(2j_2\theta_2) \sin((2j_1 + 1)\theta_1) + \frac{b-1}{\sqrt{N-1}} \cos(2j_2\theta_2) \cos((2j_1 + 1)\theta_1) \end{aligned} \quad (14)$$

This equation guarantees that the amplitude of each item in each non-target block vanishes. Now we can *measure*. In the simplest case  $N = 2^n$  and  $K = 2^k$ , so we can label blocks by  $k$  qubits [items inside of a block are labeled by  $n - k$  qubits]. We measure only  $k$  block qubits and **find the target block**. We shall choose the numbers of iterations  $j_1$  and  $j_2$  by minimizing the total number of iterations  $j_1 + j_2$ .

To see universal features we consider the limit when each block is very large  $b \rightarrow \infty$ , this makes the total number of items in the whole database also large  $N = Kb \rightarrow \infty$ . The expression for angles (7), (12) simplifies:

$\theta_1 = 1/\sqrt{N}$ ,  $\theta_2 = 1/\sqrt{b}$ . It was shown in [7] that the numbers of iterations scales as:

$$j_1 = \frac{\pi}{4}\sqrt{N} - \eta\sqrt{b}, \quad j_2 = \alpha\sqrt{b}, \quad \mathbf{c} = \eta - \alpha. \quad (15)$$

Here  $\eta$  and  $\alpha$  are parameters of order of 1 [they have a limit]. For large blocks  $b \rightarrow \infty$  the equation (14) can be simplified to:

$$\tan\left(\frac{2\eta}{\sqrt{K}}\right) = \frac{2\sqrt{K}\sin 2\alpha}{K - 4\sin^2 \alpha} \quad (16)$$

### Minimization of Total Number of Iterations.

Let us minimize the number of queries to the oracle [number of iterations]:  $S = j_1 + j_2 + 1 \rightarrow \frac{\pi}{4}\sqrt{N} - \mathbf{c}\sqrt{b}$ . Here  $\mathbf{c} = \eta - \alpha$ . To optimize the algorithm we have to minimize  $(\alpha - \eta)$  having in mind constrain (16). The author found the optimal values of  $\alpha$  and  $\eta$ , they depend on  $K$ , let us distinguish them by a subindex  $\alpha_K$  and  $\eta_K$ . The minimum number of queries is achieved at:

$$\tan \frac{2\eta_K}{\sqrt{K}} = \frac{\sqrt{3K-4}}{K-2}, \quad \cos 2\alpha_K = \frac{K-2}{2(K-1)}, \quad \mathbf{c} = \eta_K - \alpha_K. \quad (17)$$

### This describes optimal version of Grover-Radhakrishnan algorithm.

Let us study dependence on number of blocks  $K$ :  $\alpha_K$  monotonically decreases with  $K$ :

$$\alpha_2 = \frac{\pi}{4} \geq \alpha_K \geq \frac{\pi}{6} = \alpha_\infty \quad (18)$$

$$K = 2 \quad \rightarrow \quad K = \infty$$

In case of two large blocks  $K = 2$  minimization of number of queries of partial search algorithm gives:  $\alpha_2 = \pi/4$ ,  $\eta_2 = \pi/2\sqrt{2}$ . This means that for  $K = 2$  algorithm skips the global iterations and makes a full local search in each block:  $j_1 = 0$ ,  $j_2 = (\pi/4)\sqrt{b}$ . For three blocks or more  $3 \leq K$  the algorithm makes less then full search of each block [locally]. Now let us analyze the number of global iterations:

$$j_1 = \left(\frac{\pi}{4} - \frac{\eta_K}{\sqrt{K}}\right)\sqrt{N} > 0, \quad \frac{d}{dK}\left(\frac{\eta_K}{\sqrt{K}}\right) < 0, \quad \frac{dj_1}{dK} > 0, \quad \text{for } 3 \leq K. \quad (19)$$

Parameter  $\eta_K$  decreases monotonically from  $\eta_2 = \pi/(2\sqrt{2})$  to  $\eta_\infty = \sqrt{3/4}$ , when  $K$  increases.

The difference  $\alpha_K - \eta_K$  monotonically decrease with  $K$ . Numerical values of  $\alpha_K$  and  $\eta_K$  for different number of blocks are:

$$\begin{aligned} \alpha_2 &\approx 0.7854, & \eta_2 &\approx 1.1107, & \alpha_2 - \eta_2 &\approx -0.3253 \\ \alpha_3 &\approx 0.65906, & \eta_3 &\approx 0.9961, & \alpha_3 - \eta_3 &\approx -0.33704 \\ \alpha_4 &\approx 0.6155, & \eta_4 &\approx 0.9553, & \alpha_4 - \eta_4 &\approx -0.3398 \\ \alpha_5 &\approx 0.5932, & \eta_5 &\approx 0.9341, & \alpha_5 - \eta_5 &\approx -0.3409 \\ \alpha_\infty &\approx 0.5236, & \eta_\infty &\approx 0.866, & \alpha_\infty - \eta_\infty &\approx -0.3424 \end{aligned}$$

These are solutions of equation (17). These parameters define the number of iterations

$$j_1 = \frac{\pi}{4}\sqrt{N} - \eta_K\sqrt{b}, \quad j_2 = \alpha_K\sqrt{b}, \quad S_K \approx j_1 + j_2 \rightarrow \frac{\pi}{4}\sqrt{N} + (\alpha_K - \eta_K)\sqrt{b}. \quad (20)$$

We can compare this with the full search in *randomly* picked  $K - 1$  blocks, which takes

$$R_K = \frac{\pi}{4}\sqrt{\frac{K-1}{K}}\sqrt{N} \quad (21)$$

iterations, see (1). For two blocks partial search and random pick takes the same number of queries:  $R_2 = S_2 = \lceil \pi/4 \rceil \sqrt{N}/2$ . For more blocks partial search is faster:

$$\begin{aligned} R_3 &= 0.641\sqrt{N}, & S_3 &= 0.59\sqrt{N}, \\ R_4 &= 0.68\sqrt{N}, & S_4 &= 0.586\sqrt{N}, \\ R_5 &= 0.702\sqrt{N}, & S_5 &= 0.63\sqrt{N}. \end{aligned}$$

Here we compared random pick algorithm with the partial search algorithm using:  $S_K = \left( \pi/4 + \lceil \alpha_K - \eta_K \rceil / \sqrt{K} \right) \sqrt{N}$ . We see that starting from  $K = 3$  partial search algorithm works faster then random pick. As the number of blocks increases the advantage becomes more essential.

But for large  $K$  we should compare the partial search algorithm with its *interrupted* version: If we make only global iterations of the partial search algorithm and measure the wave function of the database, probability to find the target item is:

$$p_t = \sin^2((2j_1 + 1)\theta_1) = \frac{(K - 2)^2}{K(K - 1)}. \quad (22)$$

It monotonically increases with  $K$ .

Let us solve equations (17) explicitly for large  $K$ :

$$\alpha_K \rightarrow \frac{\pi}{6} + \frac{1}{2\sqrt{3}K} + \frac{5\sqrt{3}}{(6K)^2}, \quad \eta_K \rightarrow \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}K} + \frac{11\sqrt{3}}{90K^2}. \quad K \rightarrow \infty$$

Corrections to these expressions are of order  $1/K^3$ . The total number of queries is:

$$S_K \rightarrow \frac{\pi}{4}\sqrt{N} + (\alpha_K - \eta_K)\sqrt{b}, \quad -\mathbf{c} = \alpha_K - \eta_K = \frac{\pi}{6} - \sqrt{\frac{3}{4}} + \frac{1}{5\sqrt{3}(2K)^2} < 0. \quad (23)$$

Random pick (21) takes more queries:

$$R_K \rightarrow \frac{\pi}{4}\sqrt{N} - \left( \frac{\pi}{8\sqrt{K}} \right) \sqrt{b}, \quad K \rightarrow \infty. \quad (24)$$

As for the *interrupted* version of the algorithm in the limit of  $K \rightarrow \infty$ , the probability to find the target item by measuring after global iterations is close to certainty:  $p_t = 1 - 3/K$ ,  $K \rightarrow \infty$  see (22). The partial search algorithm is efficient for limited number of blocks only:  $3 \leq K \leq 3/(1 - p_t)$ . If we choose the probability  $p_t = 0.9$  then the partial search algorithm works well in the region:

$$3 \leq K \leq 30. \quad (25)$$

The version of partial search algorithm described here is little faster then original Grover-Radhakrishnan algorithm [7]: in the expression for total number of iterations  $S_K$  the coefficient  $\mathbf{c} = \eta_K - \alpha_K$  in (23) and (20) is from 1% to 3% larger [depending on  $K$ ]. But our version uses the absolute minimum of queries to the oracle.

### Lower bound

A lower bound for number of queries to the oracle was found in [7]:

$$S \geq \frac{\pi}{4}\sqrt{N} - \frac{\pi}{4}\sqrt{b}. \quad (26)$$

It is based on the lower bound for the full search [2, 6]. One can first search for the block and then for the target item in the block. We can improve the lower bound for algorithms that have the same final state for the target block. After we run partial search algorithm the wave function of the database (13) has non-zero components only in the target block. The calculations show:

$$|d\rangle = \sin \alpha_K |t\rangle + \cos \alpha_K |\text{ntt}\rangle \quad (27)$$

see (17) and (11). We can represent it as a result of application of  $j_e$  Grover iterations to uniform superposition of all basis states in the target block:

$$|d\rangle = G_2^{j_e} |s_2\rangle, \quad j_e = \frac{\alpha_K}{2} \sqrt{b} \quad (28)$$

see (8) and (9). It will take only  $\tilde{j}_{\text{full}} = (\pi/4 - \alpha_K/2) \sqrt{b}$  iterations to find the target item in the target block. We can a bound  $S$  from the following:  $S + \tilde{j}_{\text{full}} \geq \pi\sqrt{N}/4$ . Lower bound depends on number of blocks, see (18). Replacing  $\alpha_K$  by its minimum (18) we get a tighter lower bound:

$$S \geq \frac{\pi}{4} \sqrt{N} + \left( -\frac{\pi}{4} + \frac{\alpha_K}{2} \right) \sqrt{b} \geq \frac{\pi}{4} \sqrt{N} - \frac{\pi}{6} \sqrt{b}. \quad (29)$$

## SUMMARY

We optimized Grover-Radhakrishnan version of partial search. We conjecture that our version of partial search is optimal in wider class of partial search algorithms [arbitrary sequences of local and global iterations].

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